Angular Probability Distribution of Three Particles near Zero Energy Threshold

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Abstract

We study bound states of the 3–particle system in $\mathbb{R}^3$ described by the Hamiltonian $H(\lambda_n) = H_0 + v_{12} + \lambda_n(v_{13} + v_{23})$, where the particle pair $\{1, 2\}$ is critically bound, and particle pairs $\{1, 3\}$ and $\{2, 3\}$ are neither bound nor critically bound. We prove the following: if $H(\lambda_n)\psi_n = E_n\psi_n$, where $E_n \to 0$ for $\lambda_n \to \lambda_{cr}$, and besides $\lim_{n\to\infty} \int_{|\zeta|\leq R} |\psi_n(\zeta)|^2 d\zeta = 0$ for any $R > 0$, then the angular probability distribution of three particles determined by ψ_n for large n approaches the exact expression, which does not depend on pair–interactions. The result has applications in Efimov physics and in the physics of halo nuclei.

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I. INTRODUCTION

Consider the Hamiltonian of the 3–particle system in $\mathbb{R}^3$

$$H(\lambda) = H_0 + v_{12} + \lambda(v_{13} + v_{23}), \tag{1}$$

where H_0 is the kinetic energy operator with the center of mass removed, $\lambda \in \mathbb{R}_+$ is the coupling constant and none of the particle pairs has bound states. The detailed requirements on pair–potentials would be listed in Sec. III. Suppose that for a converging sequence of coupling constants $\lambda_n \to \lambda_{cr}$ there exists a sequence of bound states $\psi_n \in D(H_0)$ such that $H(\lambda_n)\psi_n = E_n\psi_n$, where $E_n < 0$, $\|\psi_n\| = 1$ and $E_n \to 0$. The question, whether the sequence ψ_n totally spreads has been recently considered in [1, 2]. In [1] it was shown that ψ_n does not spread if $H(\lambda_n)$, $H(\lambda_{cr})$ have no 2–particle subsystems that are bound or critically bound. The results of [1] were generalized to many–particle systems [2], where, in particular, the restriction on the sign of pair–potentials was removed. In [1] under certain conditions on pair–potentials it was proved that if the pair of particles $\{1, 2\}$ has a zero energy resonance and ψ_n for each n is the ground state then the sequence ψ_n totally spreads.

Here we focus again on the situation, where the pair of particles $\{1, 2\}$ has a zero energy resonance and the sequence $\psi_n(x, y)$ (*not necessarily ground states!*) totally spreads. (For the definition of Jacobi coordinates $x, y \in \mathbb{R}^3$ see [1]). By definition of total spreading $\int_{|x|^2+|y|^2\leq R} |\psi_n(x, y)|^2\, d^3x d^3y \to 0$ for each $R > 0$. Thereby, especially interesting is the angular probability distribution of three particles for large n.

Let us write the wave function in the form $\psi_n(\rho, \theta, \hat{x}, \hat{y})$, where the arguments are the so–called hyperspherical coordinates [3] $\rho := \sqrt{|x^2| + |y|^2}$, $\theta := \arctan(|y|/|x|)$, $\theta \in (0, \pi/2)$ and $\hat{x}, \hat{y}$ are unit vectors in the directions of x, y respectively. Then by definition the angular probability distribution is

$$\mathcal{D}_n(\theta, \hat{x}, \hat{y}) := \cos^2\theta \sin^2\theta \int \rho^5\, \big|\psi_n(\rho, \theta, \hat{x}, \hat{y})\big|^2 d\rho. \tag{2}$$

The normalization $\|\psi_n\| = 1$ implies that

$$\int_0^{\pi/2} d\theta \int d\Omega_x \int d\Omega_y\; \mathcal{D}_n(\theta, \hat{x}, \hat{y}) = 1, \tag{3}$$

where $\Omega_{x,y}$ are the body angles associated with the unit vectors $\hat{x}, \hat{y}$. The main result of the present paper (proved in Theorem 3) states that

$$\mathcal{D}_\infty(\theta, \hat{x}, \hat{y}) := \lim_{n\to\infty} \mathcal{D}_n(\theta, \hat{x}, \hat{y}) = \frac{1}{(4\pi)^2}\frac{4}{\pi}\sin^2\theta, \tag{4}$$

where the convergence is in measure. Equation (4) means that for all acceptable pair–potentials the limiting angular probability distribution exists and depends solely on θ. Apart from [1, 2] the proof resides on the ideas expressed in [4–7]. In the next section we shall discuss the two–particle case, this material would also be needed in the analysis of the three–particle case in Sec. III.

II. THE TWO-PARTICLE CASE REVISITED

Let us consider the two–particle Hamiltonian in $L^2(\mathbb{R}^3)$

$$h(\lambda) = -\Delta_x + \lambda v(x), \tag{5}$$

where $\lambda \in \mathbb{R}_+$ is a coupling constant. For the pair potential we assume that $\gamma < \infty$, where

$$\gamma := \max \left[\int d^3x \; |x|^2 \big(1 + |x|^\delta\big) |v(x)|^2, \int d^3x \; \big(1 + |x|^\delta\big) |v(x)|^2 \right] \tag{6}$$

and $0 < \delta < 1$ is some constant.

The next theorem (which must be known in some form) states that a totally spreading sequence of bound state wave functions approaches the expression, which is independent of the details of the pair–interaction.

Theorem 1. *Suppose there is a sequence of coupling constants $\lambda_n \in \mathbb{R}_+$ such that* $\lim_{n\to\infty} \lambda_n = \lambda_{cr} \in \mathbb{R}_+$*, and $H(\lambda_n)\psi_n = E_n \psi_n$, where $\psi_n \in D(H_0)$, $\|\psi_n\| = 1$, $E_n < 0$,* $\lim_{n\to\infty} E_n = 0$*. If ψ_n totally spreads then*

$$\left\| \psi_n - e^{i\varphi_n} \frac{\sqrt{k_n} e^{-k_n|x|}}{\sqrt{2\pi}|x|} \right\| \to 0, \tag{7}$$

where $\varphi_n \in \mathbb{R}$ are phases and $k_n := \sqrt{|E_n|}$.

A few remarks are in order. If one takes for ψ_n the ground states then the sequence totally spreads, see the discussion in [6, 8]. In the spherically symmetric potential s–states always spread, and p–states do not [6]. (This can also be seen from (7) telling that the wave function approaches the spherically symmetric form). Let us also note that ψ_n does not spread if $v(x) \geq |x|^{-2+\epsilon}$ for $|x| \geq R_0$ and $\epsilon \in (0, 1)$, see [8, 9].

Proof of Theorem 1. $R_n := (\psi_n, (1 + |x|^\delta)^{-1}\psi_n) \to 0$ because ψ_n totally spreads. The Schrödinger equation in the integral form reads

$$\tilde{\psi}_n = \frac{\lambda_n}{4\pi} \int d^3x' \; \frac{e^{-k_n|x-x'|}}{|x - x'|} v(x') \tilde{\psi}_n(x'), \tag{8}$$

where $\tilde{\psi}_n := \psi_n / R_n^{1/2}$ is the renormalized wave function . Let us set

$$f_n := \frac{\lambda_n}{4\pi} \frac{e^{-k_n|x|}}{|x|} \int d^3x' \; v(x')\tilde{\psi}_n(x'). \tag{9}$$

Our aim is to prove that $\|\tilde{\psi}_n - f_n\| = \mathcal{O}(1)$. The direct calculation gives

$$\|\tilde{\psi}_n - f_n\|^2 = \frac{\lambda_n^2}{16\pi^2} \int d^3x d^3x' d^3x'' \left[\frac{e^{-k_n|x-x'|}}{|x-x'|} - \frac{e^{-k_n|x|}}{|x|}\right] \left[\frac{e^{-k_n|x-x''|}}{|x-x''|} - \frac{e^{-k_n|x|}}{|x|}\right] \tag{10}$$

$$\times v(x')v(x'')\tilde{\psi}_n^*(x')\tilde{\psi}_n(x''). \tag{11}$$

This can be transformed into

$$\|\tilde{\psi}_n - f_n\|^2 = \frac{\lambda_n^2}{16\pi^2} \int d^3x' d^3x'' \frac{1}{k_n} \Big\{W(k_n(x''-x')) + W(0) - W(k_n x') - W(k_n x'')\Big\}$$

$$\times v(x')v(x'')\tilde{\psi}_n^*(x')\tilde{\psi}_n(x''), \tag{12}$$

where we defined

$$W(y) := \int d^3z \; \frac{e^{-|z|}e^{-|z-y|}}{|z| \; |z-y|} = 2\pi e^{-|y|}. \tag{13}$$

The integral in (13) can be evaluated using the confocal elliptical coordinates, see f. e. Appendix 9 in [10]. Next, by the obvious inequality $|W(y) - W(0)| \leq 2\pi |y|$

$$\|\tilde{\psi}_n - f_n\|^2 \leq \frac{\lambda_n^2}{8\pi} \int d^3x' d^3x'' \big\{|x''-x'| + |x'| + |x''|\big\} |v(x')||v(x'')||\tilde{\psi}_n(x')||\tilde{\psi}_n(x'')|$$

$$\leq \frac{\lambda_n^2}{2\pi} \int d^3x' d^3x'' |x'||v(x')||v(x'')||\tilde{\psi}_n(x')||\tilde{\psi}_n(x'')|. \tag{14}$$

Inserting into the rhs of (14) the identities $1 = (1+|x'|^\delta)^{1/2}(1+|x'|^\delta)^{-1/2}$ and the same for x'' and applying the Cauchy–Schwarz inequality gives

$$\|\tilde{\psi}_n - f_n\|^2 \leq \frac{\lambda_n^2 \gamma}{2\pi}, \tag{15}$$

where γ is defined in (6). Thus $\|\tilde{\psi}_n - f_n\| = \mathcal{O}(1)$ and by (9) we have

$$\psi_n = \frac{\lambda_n}{4\pi} R_n^{1/2} d_n \frac{e^{-k_n|x|}}{|x|} + \mathrm{o}(1), \tag{16}$$

where $d_n := \int d^3x' \; v(x')\tilde{\psi}_n(x')$ and $\mathrm{o}(1)$ denotes the terms that go to zero in norm. Using that $\|\psi_n\| = 1$ we recover the statement of the theorem. $\square$

III. THE THREE–PARTICLE CASE

We shall consider the Hamiltonian (1). Let m_i and $r_i \in \mathbb{R}^3$ denote particles masses and position vectors. The reduced masses we shall denote as $\mu_{ik} := m_i m_k (m_i + m_k)$. The pair–interactions v_{ik} are operators of multiplication by real $V_{ik}(r_i - r_k)$. We shall impose the following restrictions

R1 The pair potentials satisfy the following requirement

$$\gamma_0 := \max_{i=1,2} \max \left[\int d^3 r \big|V_{i3}(r)\big|^2, \int d^3 r \big|V_{i3}(r)\big|(1+|r|)^{2\delta} \right] < \infty, \tag{17}$$

where $0 < \delta < 1/8$ is a fixed constant. And

$$- b_1 e^{-b_2 |r|} \leq V_{12}(r) \leq 0, \tag{18}$$

where $b_{1,2} > 0$ are some constants.

R2 There is a sequence of coupling constants $\lambda_n \in \mathbb{R}_+$ such that $\lim_{n\to\infty} \lambda_n = \lambda_{cr} \in \mathbb{R}_+$, and $H(\lambda_n)\psi_n = E_n \psi_n$, where $\psi_n \in D(H_0)$, $\|\psi_n\| = 1$, $E_n < 0$, $\lim_{n\to\infty} E_n = 0$.

R3 The Hamiltonian $H_0 + v_{12}$ is at critical coupling (For the definition of critical coupling see [2]). The Hamiltonians $H_0 + \lambda v_{13}$ and $H_0 + \lambda v_{23}$ are positive and are not at critical coupling for $\lambda = \lambda_n, \lambda_{cr}$.

In the Jacobi coordinates $x := [\sqrt{2\mu_{12}}/\hbar](r_2 - r_1)$ and $y := [\sqrt{2M_{12}}/\hbar](r_3 - m_1/(m_1 + m_2)r_1 - m_2/(m_1+m_2)r_2)$, where $M_{ij} = (m_i + m_j)m_k/(m_1+m_2+m_3)$ ($\{i,j,k\}$ is a permutation of $\{1,2,3\}$) the kinetic energy operator takes the form [1, 2]

$$H_0 = -\Delta_x - \Delta_y. \tag{19}$$

In the following $\chi_\Omega : \mathbb{R} \to \mathbb{R}$ denotes the characteristic function of the interval $\Omega \subset \mathbb{R}$. The next theorem is the analog of Theorem 1 for the three–particle case.

Theorem 2. *Suppose $H(\lambda)$ defined in (1) satisfies R1–3. If ψ_n totally spreads then*

$$\left\| \psi_n - \frac{e^{i\varphi_n} \chi_{[1,\infty)}(\rho)}{2\pi^{3/2} |\ln k_n|^{1/2}} \frac{\big\{|x| \sin(k_n|y|) + |y|\cos(k_n|y|)\big\} e^{-k_n|x|}}{|x|^3|y| + |y|^3|x|} \right\| \to 0, \tag{20}$$

where $\varphi_n \in \mathbb{R}$ are phases, $\rho := \sqrt{|x|^2 + |y|^2}$ and $k_n := \sqrt{|E_n|}$.

This theorem has a useful practical corollary.

Theorem 3. *The angular probability distribution $\mathcal{D}_n(\theta, \hat{x}, \hat{y})$ defined in (2) converges in measure to $\mathcal{D}_\infty(\theta, \hat{x}, \hat{y}) = (4\pi^3)^{-1} \sin^2\theta$.*

Proof. Let us rewrite (20) in hyperspherical coordinates

$$\|\psi_n - \Theta_n\| \to 0, \tag{21}$$

where

$$\Theta_n := \frac{e^{i\varphi_n}\chi_{[1,\infty)}(\rho)}{2(\pi)^{3/2}|\ln k_n|^{1/2}} \frac{e^{-k_n\rho\cos\theta}\sin(\theta + k_n\rho\sin\theta)}{\rho^3\cos\theta\sin\theta}. \tag{22}$$

If we denote by $\mathcal{D}_n^\Theta(\theta, \hat{x}, \hat{y})$ the angular probability distribution given by Θ_n then the limiting angular probability distribution is

$$\mathcal{D}_\infty(\theta, \hat{x}, \hat{y}) = \lim_{n\to\infty} \mathcal{D}_n^\Theta = \frac{1}{4\pi^3} \lim_{n\to\infty} \frac{1}{|\ln k_n|} \int_1^\infty \frac{e^{-2k_n\rho\cos\theta}}{\rho} \sin^2(\theta + k_n\rho\sin\theta)\, d\rho, \tag{23}$$

where the limit is pointwise. Changing the integration variable in the last integral for $t = k_n\rho\sin\theta$ and expanding around $t = 0$ we obtain

$$\mathcal{D}_\infty(\theta, \hat{x}, \hat{y}) = \frac{1}{4\pi^3} \lim_{n\to\infty} \frac{1}{|\ln k_n|} \int_{k_n\sin\theta}^\infty \frac{e^{-2t\cot\theta}}{t} \sin^2(\theta + t)\, dt = \frac{1}{(4\pi)^2}\frac{4}{\pi}\sin^2\theta. \tag{24}$$

Note that $\mathcal{D}_n^\Theta \to \mathcal{D}_\infty$ pointwise and uniformly. Now we show that $\|\mathcal{D}_n - \mathcal{D}_n^\Theta\|_1 \to 0$. To make the notation shorter we set $d\tilde{\Omega} := \cos^2\theta\sin^2\theta d\theta d\Omega_x d\Omega_y$.

$$\|\mathcal{D}_n - \mathcal{D}_n^\Theta\|_1 \equiv \int_0^{\pi/2} d\theta \int d\Omega_x d\Omega_y \big|\mathcal{D}_n - \mathcal{D}_n^\Theta\big| = \int d\tilde{\Omega} \Big| \int |\psi_n|^2\rho^5 d\rho - \int |\Theta_n|^2 \rho^5 d\rho \Big| \tag{25}$$

$$\leq \int d\tilde{\Omega} \int \rho^5 \Big| |\psi_n| - |\Theta_n| \Big| \Big(|\psi_n| + |\Theta_n| \Big)\, d\rho \leq \int d\tilde{\Omega} \int \rho^5 \Big| \psi_n - \Theta_n \Big| \Big(|\psi_n| + |\Theta_n| \Big)\, d\rho \tag{26}$$

$$\leq \big\|\psi_n - \Theta_n\big\| \left(\int d\tilde{\Omega} \int \rho^5 \big(|\psi_n| + |\Theta_n|\big)^2 d\rho \right)^{1/2} \leq 2\big\|\psi_n - \Theta_n\big\|, \tag{27}$$

where we applied twice the Cauchy–Schwarz inequality and $\big||a| - |b|\big| \leq |a - b|$ for any $a, b \in \mathbb{C}$. Therefore, $\|\mathcal{D}_n - \mathcal{D}_\infty\|_1 \to 0$. By the Vitali convergence theorem [11] this is equivalent to the statement of the Theorem 3. □

Remark. If instead of Jacobi coordinates one would express the limiting angular probability distribution in $r_{13} := r_3 - r_1$ and $r_{23} := r_3 - r_2$, which are also "natural" coordinates for the considered problem, then it would depend not only on the ratio $|r_{13}|/|r_{23}|$ but also on the angle between these vectors. Let us also note that if the pair of particles $\{1, 2\}$ would be

marginally bound with the energy E_{12} and the sequence of ground states ψ_n would be such that $E_n < E_{12}$, $E_n \to E_{12}$ then ψ_n totally spreads, see [7]. However, in this case it is easy to show that the angular probability distribution approaches the delta–distribution.

Lemma 1. *Suppose $H(\lambda)$ defined in (1) satisfies R1–3. If ψ_n totally spreads then*

$$\psi_n = \big[H_0 + k_n^2\big]^{-1} |v_{12}| \psi_n + o(1), \tag{28}$$

where $o(1)$ denotes the terms that go to zero in norm.

Proof. Rearranging the terms in the Schrödinger equation for ψ_n we obtain three equivalent integral equations, see [2]

$$\psi_n = \big[H_0 + k_n^2\big]^{-1} \Big(|v_{12}| - \lambda_n v_{13} - \lambda_n v_{23}\Big)\psi_n, \tag{29}$$

$$\psi_n = \big[H_0 + \lambda_n (v_{13})_+ + \lambda_n (v_{23})_+ + k_n^2\big]^{-1} \Big(|v_{12}| + \lambda_n (v_{13})_- + \lambda_n (v_{23})_-\Big)\psi_n, \tag{30}$$

$$\psi_n = \big[H_0 + \lambda_n (v_{13})_+ + k_n^2\big]^{-1} \Big(|v_{12}| + \lambda_n (v_{13})_- - \lambda_n v_{23}\Big)\psi_n, \tag{31}$$

where $(v_{ik})_\pm = \max[0, \pm v_{ik}]$. By (29) the Lemma would be proved if we can show that

$$F_n := \lambda_n \big[H_0 + k_n^2\big]^{-1} v_{13} \psi_n = \mathrm{o}(1), \tag{32}$$

$$\lambda_n \big[H_0 + k_n^2\big]^{-1} v_{23} \psi_n = \mathrm{o}(1). \tag{33}$$

Below we prove (32), eq. (33) is proved analogously. Substituting (30) into (32) we split F_n in three parts

$$F_n = \sum_{i=1}^{3} F_n^{(i)}, \tag{34}$$

where

$$F_n^{(1)} = \big[H_0 + k_n^2\big]^{-1} v_{13} \big[H_0 + \lambda_n (v_{13})_+ + \lambda_n (v_{23})_+ + k_n^2\big]^{-1} |v_{12}| \psi_n, \tag{35}$$

$$F_n^{(2)} = \lambda_n \big[H_0 + k_n^2\big]^{-1} v_{13} \big[H_0 + \lambda_n (v_{13})_+ + \lambda_n (v_{23})_+ + k_n^2\big]^{-1} (v_{23})_- \psi_n, \tag{36}$$

$$F_n^{(3)} = \lambda_n \big[H_0 + k_n^2\big]^{-1} v_{13} \big[H_0 + \lambda_n (v_{13})_+ + \lambda_n (v_{23})_+ + k_n^2\big]^{-1} (v_{13})_- \psi_n. \tag{37}$$

We introduce another pair of Jacobi coordinates $\eta = [\sqrt{2\mu_{13}}/\hbar](r_3 - r_1)$ and $\zeta = [\sqrt{2M_{13}}/\hbar](r_2 - m_1/(m_1 + m_3) r_1 - m_3/(m_1 + m_3) r_3)$. The coordinates (η, ζ) and (x, y) are connected through the linear transformation

$$x = m_{x\eta}\eta + m_{x\zeta}\zeta, \tag{38}$$

$$y = m_{y\eta}\eta + m_{y\zeta}\zeta, \tag{39}$$

where $m_{x\eta}, m_{x\zeta} \neq 0, m_{y\eta}, m_{y\zeta}$ form the orthogonal matrix and can be expressed through mass ratios in the system. $\mathcal{F}_{13}$ denotes the partial Fourier transform, which acts on $f(\eta, \zeta)$ as

$$\mathcal{F}_{13} f := \hat{f}(\eta, p_\zeta) = \frac{1}{(2\pi)^{3/2}} \int d^3\zeta \; e^{-ip_\zeta \cdot \zeta} f(\eta, \zeta). \tag{40}$$

Let us introduce the operator function

$$\tilde{B}_{13}(k_n) := \mathcal{F}_{13}^{-1} \tilde{t}_n(p_\zeta) \mathcal{F}_{13}, \tag{41}$$

where

$$\tilde{t}_n(p_\zeta) = \begin{cases} |p_\zeta|^{1-\delta} + (k_n)^{1-\delta} & \text{if} \;\; |p_\zeta| \leq 1 \\ 1 + (k_n)^{1-\delta} & \text{if} \;\; |p_\zeta| \geq 1. \end{cases} \tag{42}$$

We set tilde over the operator in order to distinguish it from the one defined in Eq. (18) in [1]. Note that $\tilde{B}_{13}(k_n)$ and $\tilde{B}_{13}^{-1}(k_n)$ for each n are bounded operators.

Using the inequalities from [2] (see Eqs. (17)–(24) in [2]) we obtain

$$|F_n^{(1)}| \leq \big[H_0 + k_n^2\big]^{-1} |v_{13}| \big[H_0 + k_n^2\big]^{-1} |v_{12}||\psi_n| = \big[H_0 + k_n^2\big]^{-1} |v_{13}|^{1/2} \tilde{B}_{13}(k_n) \Psi_n^{(1)}, \tag{43}$$

$$|F_n^{(2)}| \leq \lambda_n \big[H_0 + k_n^2\big]^{-1} |v_{13}| \big[H_0 + k_n^2\big]^{-1} |v_{23}||\psi_n| = \big[H_0 + k_n^2\big]^{-1} |v_{13}|^{1/2} \tilde{B}_{13}(k_n) \Psi_n^{(2)}, \tag{44}$$

where

$$\Psi_n^{(1)} := |v_{13}|^{1/2} \tilde{B}_{13}^{-1}(k_n) \big[H_0 + k_n^2\big]^{-1} |v_{12}||\psi_n|, \tag{45}$$

$$\Psi_n^{(2)} := \lambda_n |v_{13}|^{1/2} \tilde{B}_{13}^{-1}(k_n) \big[H_0 + k_n^2\big]^{-1} |v_{23}||\psi_n|. \tag{46}$$

To write the bound on $|F_n^{(3)}|$ we use the following expression, which follows from (31), c. f. Eq. (15) in [2]

$$(v_{13})_-^{1/2} \psi_n = Q_n (v_{13})_-^{1/2} \big[H_0 + \lambda_n (v_{13})_+ + k_n^2\big]^{-1} \big(|v_{12}| - \lambda_n v_{23}\big) \psi_n, \tag{47}$$

where we defined

$$Q_n := \Big\{1 - \lambda_n (v_{13})_-^{1/2} \big[H_0 + \lambda_n (v_{13})_+ + k_n^2\big]^{-1} (v_{13})_-^{1/2} \Big\}^{-1}. \tag{48}$$

Q_n is a positivity preserving and uniformly norm–bounded operator, see Lemma 1 in [2]. Substituting (47) into (37) and using the positivity preserving property of the operators (see the discussion after Eq. (16) in [2]) we get

$$\begin{aligned} |F_n^{(3)}| &\leq \lambda_n \big[H_0 + k_n^2\big]^{-1} |v_{13}| \big[H_0 + k_n^2\big]^{-1} (v_{13})_-^{1/2} Q_n (v_{13})_-^{1/2} \big[H_0 + k_n^2\big]^{-1} \big(|v_{12}| + \lambda_n |v_{23}|\big) |\psi_n| \\ &= \big[H_0 + k_n^2\big]^{-1} |v_{13}|^{1/2} \tilde{B}_{13}(k_n) \Psi_n^{(3)}, \end{aligned} \tag{49}$$

where

$$\Psi_n^{(3)} := \lambda_n |v_{13}|^{1/2}\big[H_0 + k_n^2\big]^{-1}(v_{13})_-^{1/2} Q_n \tilde{B}_{13}^{-1}(k_n)(v_{13})_-^{1/2}\big[H_0 + k_n^2\big]^{-1}\big(|v_{12}| + \lambda_n |v_{23}|\big)|\psi_n|. \tag{50}$$

Summarizing, (43), (44) and (49) can be expressed through the inequality

$$|F_n^{(i)}| \leq \mathcal{L}_n \Psi_n^{(i)} \qquad (i = 1, 2, 3), \tag{51}$$

where

$$\mathcal{L}_n := \big[H_0 + k_n^2\big]^{-1}|v_{13}|^{1/2}\tilde{B}_{13}(k_n). \tag{52}$$

From Lemma 2 it follows that $\|F_n^{(i)}\| \to 0$. □

Lemma 2. *The operators $\mathcal{L}_n$ are uniformly norm–bounded and $\|\Psi_n^{(i)}\| \to 0$ for $i = 1, 2, 3$.*

Proof. The proof that $\mathcal{L}_n$ are uniformly norm–bounded is similar to Lemma 6 in [1]. The operator $K_n := \mathcal{F}_{13}\mathcal{L}_n\mathcal{F}_{13}^{-1}$ with the kernel

$$k_n(\eta, \eta'; p_\zeta) = \frac{e^{-\sqrt{p_\zeta^2 + k_n^2}|\eta - \eta'|}}{4\pi|\eta - \eta'|}\big|V_{13}(\alpha'\eta')\big|^{1/2}\tilde{t}_n(p_\zeta), \tag{53}$$

where $\alpha' := \hbar/\sqrt{2\mu_{13}}$, acts on $f(\eta, p_\zeta) \in L^2(\mathbb{R}^6)$ as

$$K_n f = \int d^3\eta' k_n(\eta, \eta'; p_\zeta) f(\eta', p_\zeta). \tag{54}$$

So, we can estimate the norm as follows

$$\|\mathcal{L}_n\|^2 = \|K_n\|^2 \leq \sup_{p_\zeta} \int \big|k_n(\eta, \eta'; p_\zeta)\big|^2 \, d^3\eta' d^3\eta = C_0 \sup_{p_\zeta} \frac{|\tilde{t}_n(p_\zeta)|^2}{\sqrt{p_\zeta^2 + k_n^2}}, \tag{55}$$

where

$$C_0 := \frac{1}{16\pi^2}\left(\int \frac{e^{-2|s|}}{|s|^2} d^3 s\right)\left(\int \big|V_{13}(\alpha'\eta)\big| d^3\eta\right) \leq \frac{\gamma_0}{8\pi}, \tag{56}$$

where γ_0 was defined in (17). Substituting (42) into (55) it is easy to see that $\|\mathcal{L}_n\|$ is uniformly bounded. Let us rewrite (45) as

$$\Psi_n^{(1)} := \big[\mathcal{M}_n^{(1)} + \mathcal{M}_n^{(2)}\big]|v_{12}|^{1/2}|\psi_n|, \tag{57}$$

where

$$\mathcal{M}_n^{(1)} := |v_{13}|^{1/2}\Big\{\tilde{B}_{13}^{-1}(k_n) - \big(1 + (k_n)^{1-\delta}\big)^{-1}\Big\}\big[H_0 + k_n^2\big]^{-1}|v_{12}|^{1/2}, \tag{58}$$

$$\mathcal{M}_n^{(2)} := \big(1 + (k_n)^{1-\delta}\big)^{-1}|v_{13}|^{1/2}\big[H_0 + k_n^2\big]^{-1}|v_{12}|^{1/2}. \tag{59}$$

By the no–clustering theorem $\big\||v_{12}|^{1/2}|\psi_n|\big\| \to 0$, see Appendix in [2]. Thus to prove that $\big\|\Psi_n^{(1)}\big\| \to 0$ it is enough to show that $\mathcal{M}_n^{(1,2)}$ are uniformly norm–bounded. It is easy to see that $\|\mathcal{M}_n^{(2)}\|$ is uniformly norm–bounded, see f. e. the proof of Lemma 7 in [1]. Next, $\|\mathcal{M}_n^{(1)}\| = \|K_n'\|$, where $K_n' := \mathcal{F}_{13}\mathcal{M}_n\mathcal{F}_{13}^{-1}$ is the integral operator with the kernel

$$
\begin{aligned}
k_n'(\eta, \eta', p_\zeta, p_\zeta') = \frac{1}{2^{7/2}\pi^{5/2}\omega^3}\Big[\tilde{t}_n^{-1}(p_\zeta) - \big(1 + (k_n)^{1-\delta}\big)^{-1}\Big]\big|V_{13}(\alpha'\eta)\big|^{1/2} \\
\times \frac{e^{-\sqrt{p_\zeta^2 + k_n^2}|\eta - \eta'|}}{|\eta - \eta'|} \exp\Big\{ i\frac{\beta}{\omega}\eta' \cdot (p_\zeta - p_\zeta')\Big\} \widehat{\big|V_{12}\big|^{1/2}}\big((p_\zeta - p_\zeta')/\omega\big),
\end{aligned}
\tag{60}
$$

$\beta := -m_3\hbar/((m_1 + m_3)\sqrt{2\mu_{13}})$ and $\omega := \hbar/\sqrt{2M_{13}}$ (see the proof of Lemma 9 in [1]). In (60) $\widehat{\big|V_{12}\big|^{1/2}}$ denotes merely the Fourier transform of $\big|V_{12}\big|^{1/2} \in L^2(\mathbb{R}^3)$. Calculation of the Hilbert–Schmidt norm gives

$$
\|\mathcal{M}_n^{(1)}\|^2 \leq \frac{C_0'}{8\omega^3\pi^3}\int_{|p_\zeta|\leq 1} \frac{\big[|p_\zeta|^{1-\delta} + (k_n)^{1-\delta}\big]^{-2}}{\sqrt{p_\zeta^2 + k_n^2}} d^3p_\zeta \leq \frac{C_0'}{8\omega^3\pi^3}\int_{|p_\zeta|\leq 1}\frac{d^3p_\zeta}{|p_\zeta|^{3-2\delta}}, \tag{61}
$$

where

$$
C_0' := C_0 \int d^3s \Big|\widehat{\big|V_{12}\big|^{1/2}}(s)\Big|^2. \tag{62}
$$

From (61) it follows that $\|\mathcal{M}_n^{(1)}\|$ is uniformly bounded and, therefore, $\|\Psi_n^{(1)}\| \to 0$. The fact that $\|\Psi_n^{(2)}\| \to 0$ is proved analogously. To prove that $\|\Psi_n^{(3)}\| \to 0$ let us look at (50). We can write

$$
\Psi_n^{(3)} = \lambda_n \mathcal{T}_n^{(1)} Q_n \big(\mathcal{T}_n^{(2)}|v_{12}|^{1/2}|\psi_n| + \mathcal{T}_n^{(3)}|v_{23}|^{1/2}|\psi_n|\big), \tag{63}
$$

where we defined the operators

$$
\mathcal{T}_n^{(1)} := |v_{13}|^{1/2}\big[H_0 + k_n^2\big]^{-1}(v_{13})_-^{1/2}, \tag{64}
$$

$$
\mathcal{T}_n^{(2)} := \tilde{B}_{13}^{-1}(k_n)(v_{13})_-^{1/2}\big[H_0 + k_n^2\big]^{-1}|v_{12}|^{1/2}, \tag{65}
$$

$$
\mathcal{T}_n^{(3)} := \lambda_n\tilde{B}_{13}^{-1}(k_n)(v_{13})_-^{1/2}\big[H_0 + k_n^2\big]^{-1}|v_{23}|^{1/2}. \tag{66}
$$

The operators Q_n are uniformly bounded. The operators $\mathcal{T}_n^{(1)}$ are also uniformly norm–bounded. Note that $\mathcal{T}_n^{(2)} = \mathcal{M'}_n^{(1)} + \mathcal{M'}_n^{(2)}$, where $\mathcal{M'}_n^{(1,2)}$ is defined exactly as $\mathcal{M}_n^{(1,2)}$ except that $|v_{13}|$ gets replaced with $(v_{13})_-$. Hence, from the above analysis it follows that $\|\mathcal{T}_n^{(2)}\|$ is uniformly bounded. By similar arguments $\|\mathcal{T}_n^{(3)}\|$ is uniformly bounded. Thus due to $\big\||v_{ik}|^{1/2}|\psi_n|\big\| \to 0$ (see the no–clustering theorem in [2]) the expression on the rhs of (63) goes to zero in norm. □

Proof of Theorem 2. Instead of (20) we shall prove

$$\left\| \hat{\psi}_n - \frac{\sqrt{2}e^{i\varphi_n}}{4\pi |\ln k_n|^{1/2}} \frac{\chi_{[k_n,1]}(|p_y|)e^{-|p_y||x|}}{|x||p_y|} \right\| \to 0, \tag{67}$$

where the hat denotes the action of the partial Fourier transform $\mathcal{F}_{12}$, see Eq. (17) in [1]. (20) follows directly from (67) after computing explicitly the inverse Fourier transform and dropping those terms, whose norm goes to zero.

By Lemma 1 $\|\psi_n - f_n^{(1)}\| \to 0$, where we have set $f_n^{(1)} := \big[H_0 + k_n^2\big]^{-1}|v_{12}|\psi_n$. From the Schrödinger equation for the term $\sqrt{|v_{12}|}\psi_n$ we obtain

$$\sqrt{|v_{12}|}\psi_n = -\Big\{1 - \sqrt{|v_{12}|}\big(H_0 + k_n^2\big)^{-1}\sqrt{|v_{12}|}\Big\}^{-1}\sqrt{|v_{12}|}\big[H_0 + k_n^2\big]^{-1}\big(\lambda_n v_{13} + \lambda_n v_{23}\big)\psi_n. \tag{68}$$

Substituting (68) into the expression for $f_n^{(1)}$ results in

$$f_n^{(1)} = \big[H_0 + k_n^2\big]^{-1}\sqrt{|v_{12}|}\Big\{1 - \sqrt{|v_{12}|}\big(H_0 + k_n^2\big)^{-1}\sqrt{|v_{12}|}\Big\}^{-1}\Phi_n, \tag{69}$$

where

$$\Phi_n := -\lambda_n\sqrt{|v_{12}|}\big[H_0 + k_n^2\big]^{-1}\big(v_{13} + v_{23}\big)\psi_n. \tag{70}$$

From the proofs of Lemmas 6, 9 in [1] it follows that the operators $\sqrt{|v_{12}|}\big[H_0 + k_n^2\big]^{-1}\sqrt{|v_{s3}|}$ and $B_{12}^{-1}(k_n)\sqrt{|v_{12}|}\big[H_0 + k_n^2\big]^{-1}\sqrt{|v_{s3}|}$, where $B_{12}(k_n)$ is defined in Eqs. (18)–(19) in [1], are uniformly norm–bounded for $s = 1, 2$. Thus by (70) and Theorem 3 in [2] $\|\Phi_n\| \to 0$ and $\|B_{12}^{-1}(k_n)\Phi_n\| \to 0$. Acting with $\mathcal{F}_{12}$ on (69) gives

$$\hat{f}_n^{(1)} = \big[-\Delta_x + p_y^2 + k_n^2\big]^{-1}\sqrt{|v_{12}|}\Big\{1 - \sqrt{|v_{12}|}\big(-\Delta_x + p_y^2 + k_n^2\big)^{-1}\sqrt{|v_{12}|}\Big\}^{-1}\hat{\Phi}_n. \tag{71}$$

Because $\|\hat{\Phi}_n\| \to 0$ we can write

$$\hat{f}_n^{(1)} = \hat{f}_n^{(2)} + \mathrm{o}(1), \tag{72}$$

where

$$\hat{f}_n^{(2)} := \chi_{[0,\rho_0]}\left(\sqrt{p_y^2 + k_n^2}\right)\hat{f}_n^{(1)}, \tag{73}$$

and ρ_0 is a constant defined in Lemma 11 in [1]. Now using Lemma 11 in [1] (see also discussion around Eq. (111) in [1]) we obtain

$$\hat{f}_n^{(2)} = \hat{f}_n^{(3)} + \chi_{[0,\rho_0]}\left(\sqrt{p_y^2 + k_n^2}\right)\mathcal{F}_{12}\mathcal{A}_{12}(k_n)\mathcal{F}_{12}^{-1}\mathcal{Z}\left(\sqrt{p_y^2 + k_n^2}\right)B_{12}^{-1}(k_n)\hat{\Phi}_n, \tag{74}$$

where $\mathcal{A}_{12}(k_n) := \left[H_0 + k_n^2\right]^{-1}\sqrt{|v_{12}|}B_{12}(k_n)$ and $\mathcal{Z}$ defined in [1] remain uniformly norm–bounded for all n, see Lemmas 6, 11 in [1]. The function $\hat{f}_n^{(3)}$ has the form

$$\hat{f}_n^{(3)} := \chi_{[0,\rho_0]}\Big(\sqrt{p_y^2 + k_n^2}\Big)\left[-\Delta_x + |p_y|^2 + k_n^2\right]^{-1}\frac{\sqrt{|v_{12}|}}{a\sqrt{|p_y|^2 + k_n^2}}\mathbb{P}_0\hat{\Phi}_n, \tag{75}$$

where a and $\mathbb{P}_0$ are defined in Eq. (80) and Lemma 11 in [1]. Therefore, since $\|B_{12}^{-1}(k_n)\Phi_n\| \to 0$

$$\hat{f}_n^{(2)} = \hat{f}_n^{(3)} + \mathrm{o}(1). \tag{76}$$

It makes sense to introduce

$$g_n(y) := \int d^3x \phi_0(x)\Phi_n(x,y), \tag{77}$$

where ϕ_0 was defined in Eq. 77 in [1]. The following inequality trivially follows from the exponential bound on V_{12} and the definition of ϕ_0

$$\phi_0(x) \leq b_1' e^{-b_2'|x|}, \tag{78}$$

where $b_{1,2}' > 0$ are constants. From the pointwise exponential fall off of ψ_n it follows that $g_n \in L^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ for each n. We rewrite (75) with the help of (77)

$$\hat{f}_n^{(3)} = \chi_{[0,\rho_0]}\Big(\sqrt{p_y^2 + k_n^2}\Big)\left[-\Delta_x + p_y^2 + k_n^2\right]^{-1}\frac{\sqrt{|v_{12}|}\phi_0(x)\hat{g}_n(p_y)}{a\sqrt{p_y^2 + k_n^2}} \tag{79}$$

$$= \chi_{[0,\rho_0]}\Big(\sqrt{p_y^2 + k_n^2}\Big)\frac{\hat{g}_n(p_y)}{4\pi a\sqrt{p_y^2 + k_n^2}}\int d^3x' \frac{e^{-\sqrt{p_y^2+k_n^2}|x-x'|}}{|x-x'|}\phi_0(x')|V_{12}(\alpha x')|^{1/2}, \tag{80}$$

where $\alpha := \hbar/\sqrt{2\mu_{12}}$. Next, we define

$$\hat{f}_n^{(4)} := \chi_{[0,\rho_0]}\Big(\sqrt{p_y^2 + k_n^2}\Big)\frac{\hat{g}_n(0)}{4\pi a\sqrt{p_y^2 + k_n^2}}\int d^3x' \frac{e^{-\sqrt{p_y^2+k_n^2}|x-x'|}}{|x-x'|}\phi_0(x')|V_{12}(\alpha x')|^{1/2}, \tag{81}$$

where $\hat{g}_n(0) \in \mathbb{C}$ is well-defined since $g_n \in L^1(\mathbb{R}^3)$ for each n. Using Lemma 3 and the notation in (13) gives

$$\begin{aligned}\|\hat{f}_n^{(4)} - \hat{f}_n^{(3)}\|^2 &\leq \int d^3p_y\, \chi_{[0,\rho_0]}\Big(\sqrt{p_y^2 + k_n^2}\Big)\frac{c_n^2|p_y|^{2\delta}}{16\pi^2a^2(p_y^2+k_n^2)^{3/2}} \\ \times \int d^3x' \int d^3x'' W\Big(\sqrt{p_y^2+k_n^2}(x''-x')\Big)&\phi_0(x')|V_{12}(\alpha x')|^{1/2}\phi_0(x'')|V_{12}(\alpha x'')|^{1/2} \\ &\leq \frac{\vartheta^2 c_n^2}{16\pi^2a^2}\int d^3p_y\, \chi_{[0,\rho_0]}\Big(\sqrt{p_y^2 + k_n^2}\Big)\frac{|p_y|^{2\delta}}{(p_y^2+k_n^2)^{3/2}},\end{aligned} \tag{82}$$

where we used $W(s) \leq 2\pi$ and set

$$\vartheta := \int d^3x' \phi_0(x')|V_{12}(\alpha x')|^{1/2}. \tag{83}$$

The constant in (83) is bounded, hence, by Lemma 4 $\|\hat{f}_n^{(4)} - \hat{f}_n^{(3)}\| \to 0$. As the next step we introduce

$$\hat{f}_n^{(5)} := \chi_{[0,\rho_0]}\Big(\sqrt{p_y^2+k_n^2}\Big) \frac{R\Big(\sqrt{p_y^2+k_n^2}\Big)\hat{g}_n(0)}{4\pi a\sqrt{p_y^2+k_n^2}} \frac{e^{-\sqrt{p_y^2+k_n^2}|x|}}{|x|}, \tag{84}$$

where

$$R(s) := \int d^3x' \frac{e^{-s|x'|}}{|x'|}\phi_0(x')|V_{12}(\alpha x')|^{1/2}. \tag{85}$$

Like in the proof of Theorem 1 we evaluate the square of the norm of the difference

$$\begin{aligned}
\|\hat{f}_n^{(5)} - \hat{f}_n^{(4)}\|^2 \leq \int d^3p_y\, \chi_{[0,\rho_0]}\Big(\sqrt{p_y^2+k_n^2}\Big) \frac{|\hat{g}_n(0)|^2}{16\pi^2 a^2 (p_y^2+k_n^2)^{3/2}} \\
\times \int d^3x' \int d^3x'' \Big\{W\Big(\sqrt{p_y^2+k_n^2}(x''-x')\Big) + W(0) - W\Big(\sqrt{p_y^2+k_n^2}x'\Big) - W\Big(\sqrt{p_y^2+k_n^2}x''\Big)\Big\} \\
\times \phi_0(x')|V_{12}(\alpha x')|^{1/2}\phi_0(x'')|V_{12}(\alpha x'')|^{1/2} \\
\leq \frac{|\hat{g}_n(0)|^2}{2\pi a^2}\int d^3p_y\, \frac{\chi_{[0,\rho_0]}\Big(\sqrt{p_y^2+k_n^2}\Big)}{(p_y^2+k_n^2)} \\
\times \int d^3x' \int d^3x''\, |x'|\phi_0(x')|V_{12}(\alpha x')|^{1/2}\phi_0(x'')|V_{12}(\alpha x'')|^{1/2}.
\end{aligned} \tag{86}$$

On account of R1 and (78) we conclude that $\|\hat{f}_n^{(5)} - \hat{f}_n^{(4)}\| \to 0$ since $|\hat{g}_n(0)| \to 0$ by Lemma 4. Observe that

$$|R(s) - R(0)| \leq s\vartheta, \tag{87}$$

where ϑ is defined in (83). Therefore, $\|\hat{f}_n^{(6)} - \hat{f}_n^{(5)}\| \to 0$, where by definition

$$\hat{f}_n^{(6)} := \chi_{[0,\rho_0]}\Big(\sqrt{p_y^2+k_n^2}\Big) \frac{R(0)\hat{g}_n(0)}{4\pi a\sqrt{p_y^2+k_n^2}} \frac{e^{-\sqrt{p_y^2+k_n^2}|x|}}{|x|}. \tag{88}$$

Simplifying the argument of the exponential function we define

$$\hat{f}_n^{(7)} := \chi_{[0,\rho_0]}\Big(\sqrt{p_y^2+k_n^2}\Big) \frac{R(0)\hat{g}_n(0)}{4\pi a\sqrt{p_y^2+k_n^2}} \frac{e^{-|p_y||x|}}{|x|}. \tag{89}$$

After straightforward calculation we obtain

$$\begin{aligned}
\|\hat{f}_n^{(7)} - \hat{f}_n^{(6)}\|^2 = \int d^3p_y\, \chi_{[0,\rho_0]}\Big(\sqrt{p_y^2+k_n^2}\Big) \frac{R^2(0)|\hat{g}_n(0)|^2}{4\pi a^2(p_y^2+k_n^2)} \\
\times \left[\frac{1}{2\sqrt{p_y^2+k_n^2}} + \frac{1}{2|p_y|} - \frac{2}{\sqrt{p_y^2+k_n^2}+|p_y|}\right].
\end{aligned} \tag{90}$$

Replacing in the last fraction $|p_y|$ with $\sqrt{p_y^2+k_n^2}$ results in the following inequality

$$\|\hat{f}_n^{(7)} - \hat{f}_n^{(6)}\|^2 \leq \frac{R^2(0)|\hat{g}_n(0)|^2}{8\pi a^2} \int d^3p_y \, \frac{\chi_{[0,\rho_0]}\Big(\sqrt{p_y^2+k_n^2}\Big)}{(p_y^2+k_n^2)} \left[\frac{1}{|p_y|} - \frac{1}{\sqrt{p_y^2+k_n^2}}\right]. \tag{91}$$

The integrals can be calculated explicitly, see [12], which results in $\|\hat{f}_n^{(7)} - \hat{f}_n^{(6)}\| \to 0$. At last, we simplify the expression setting

$$\hat{f}_n^{(8)} := \frac{R(0)\hat{g}_n(0)}{4\pi a} \frac{\chi_{[k_n,1]}\big(|p_y|\big)e^{-|p_y||x|}}{|x||p_y|}. \tag{92}$$

Again, one easily finds that $\|\hat{f}_n^{(8)} - \hat{f}_n^{(7)}\| \to 0$. Summarizing, we have $\|\hat{f}_n^{(i+1)} - \hat{f}_n^{(i)}\| \to 0$ for $i = 1, \ldots, 7$. Thus from $\|\hat{\psi}_n - \hat{f}_n^{(1)}\| \to 0$ it follows that $\|\hat{\psi}_n - \hat{f}_n^{(8)}\| \to 0$. Using that $\|\hat{\psi}_n\| = 1$ we obtain (67). $\square$

Lemma 3. *There exists a sequence $c_n \in \mathbb{R}_+$, $c_n \to 0$ such that*

$$\big|\hat{g}_n(p_y) - \hat{g}_n(0)\big| \leq c_n |p_y|^\delta, \tag{93}$$

where δ is defined in (17).

Proof. The trivial inequality $|e^{ip_y\cdot y} - 1| \leq |p_y|^\delta |y|^\delta$ implies that

$$\big|\hat{g}_n(p_y) - \hat{g}_n(0)\big| \leq \int d^3y \big|e^{ip_y\cdot y} - 1\big||g_n(y)| \leq |p_y|^\delta c_n, \tag{94}$$

where $c_n = \int d^3y |y|^\delta |g_n(y)|$ goes to zero by Lemma 4. $\square$

The following lemma makes use of the absence of zero energy resonances in particle pairs $\{1,3\}$ and $\{2,3\}$.

Lemma 4. *The sequence $c_n = \int d^3y \big(1 + |y|^\delta\big)|g_n(y)|$ is well-defined and goes to zero.*

Proof. By definitions (77) and (70) we have $|g_n(y)| \leq |g_n^{(1)}(y)| + |g_n^{(2)}(y)|$, where

$$g_n^{(1)}(y) := \lambda_n \int d^3x \, \phi_0 |v_{12}|^{1/2} \big[H_0 + k_n^2\big]^{-1} v_{13}\psi_n, \tag{95}$$

$$g_n^{(2)}(y) := \lambda_n \int d^3x \, \phi_0 |v_{12}|^{1/2} \big[H_0 + k_n^2\big]^{-1} v_{23}\psi_n. \tag{96}$$

Consequently $c_n \leq c_n^{(1)} + c_n^{(2)}$, where

$$c_n^{(i)} := \int d^3y \big(1 + |y|^\delta\big)|g_n^{(i)}(y)|. \tag{97}$$

Below we shall prove that $c_n^{(1)} \to 0$, the fact that $c_n^{(2)} \to 0$ is proved analogously. Let us mention that appearing integrals and interchanged oder of integration can be easily justified using the pointwise exponential fall off of ψ_n [13].

We have

$$|g_n^{(1)}(y)| \leq \int d^3x \left|V_{12}(\alpha x)\right|^{1/2} \phi_0(x) |F_n|(x,y), \tag{98}$$

where F_n was defined in (34). On account of R1 and (78) it follows that

$$|g_n^{(1)}(y)| \leq \tilde{b}_1 \int d^3x e^{-\tilde{b}_2|x|} |F_n|(x,y), \tag{99}$$

where $\tilde{b}_{1,2} > 0$ are constants. Using (35)–(37) gives

$$|F_n| \leq \sum_{i=1}^{3} |F_n^{(i)}| \leq \sum_{i=1}^{3} \tilde{F}_n^{(i)}, \tag{100}$$

$$\tilde{F}_n^{(i)} := \left[H_0 + k_n^2\right]^{-1} |v_{13}|^{1/2} \tilde{B}_{13}(k_n) \Psi_n^{(i)}. \tag{101}$$

Substituting (99), (100) into (97) we obtain

$$c_n^{(1)} \leq \tilde{b}_1 \sum_{i=1}^{3} \int d^3\eta\, d^3\zeta \left(1 + \left|m_{y\eta}\eta + m_{y\zeta}\zeta\right|^{\delta}\right) e^{-\tilde{b}_2|m_{x\eta}\eta + m_{x\zeta}\zeta|} \tilde{F}_n^{(i)}(\eta, \zeta). \tag{102}$$

Let us consider the term $\tilde{F}_n^{(i)}(\eta,\zeta)$. Denoting the integral kernel of $\left[H_0 + k_n^2\right]^{-1}$ by $G_n(\eta - \eta', \zeta - \zeta')$ from (101) we get

$$\tilde{F}_n^{(i)}(\eta,\zeta) = \int d^3\eta' |V_{13}(\alpha'\eta')|^{1/2} \int d^3\zeta'\, G_n(\eta - \eta', \zeta - \zeta') \left[\tilde{B}_{13}(k_n)\Psi_n^{(i)}\right](\eta', \zeta'). \tag{103}$$

Applying to the inner convolution integral the direct and inverse partial Fourier transforms (40) we can rewrite (103) as

$$\tilde{F}_n^{(i)}(\eta,\zeta) = \frac{1}{32\pi^4} \int d^3\eta' d^3p_\zeta\, e^{ip_\zeta\cdot\zeta} \left|V_{13}(\alpha'\eta')\right|^{1/2} \frac{e^{-\sqrt{p_\zeta^2 + k_n^2}|\eta-\eta'|}}{|\eta - \eta'|} \tilde{t}_n(p_\zeta) \hat{\Psi}_n^{(i)}(\eta', p_\zeta). \tag{104}$$

Hence,

$$\left|\tilde{F}_n^{(i)}(\eta,\zeta)\right| \leq \frac{1}{32\pi^4} \int d^3\eta' d^3p_\zeta \left|V_{13}(\alpha'\eta')\right|^{1/2} \frac{e^{-\sqrt{p_\zeta^2 + k_n^2}|\eta-\eta'|}}{|\eta - \eta'|} \tilde{t}_n(p_\zeta) \left|\hat{\Psi}_n^{(i)}(\eta', p_\zeta)\right|. \tag{105}$$

Substituting (105) into (102) and interchanging the order of integration we obtain the inequality

$$c_n^{(1)} \leq \frac{\tilde{b}_1}{32\pi^4} \sum_{i=1}^{3} \int d^3\eta' \int d^3p_\zeta \left|V_{13}(\alpha'\eta')\right|^{1/2} \tilde{t}_n(p_\zeta) \left|\hat{\Psi}_n^{(i)}(\eta', p_\zeta)\right| J(\eta', p_\zeta), \tag{106}$$

where we define

$$J(\eta', p_\zeta) := \int d^3\eta \int d^3\zeta \; \frac{e^{-\sqrt{p_\zeta^2+k_n^2}|\eta-\eta'|}}{|\eta-\eta'|} \Big(1 + \big|m_{y\eta}\eta + m_{y\zeta}\zeta\big|^\delta\Big) e^{-\tilde{b}_2|m_{x\eta}\eta+m_{x\zeta}\zeta|}. \tag{107}$$

Applying the Cauchy–Schwarz inequality to (106) results in

$$c_n^{(1)} \leq \frac{\tilde{b}_1}{32\pi^4} \sum_{i=1}^{3} \big\|\Psi_n^{(i)}\big\| \left(\int d^3\eta' \int d^3p_\zeta \; \big|V_{13}(\alpha'\eta')\big| \, \tilde{t}_n^{\,2}(p_\zeta)\, J^2(\eta', p_\zeta) \right)^{1/2}. \tag{108}$$

Inserting the estimate from Lemma 5 we finally get

$$\begin{aligned} c_n^{(1)} \leq \frac{\tilde{b}_1 c\sqrt{C\pi}}{16\pi^4} \sum_{i=1}^{3} \big\|\Psi_n^{(i)}\big\| \Bigg(\int_0^1 \frac{s^2\big(s^{1-\delta}+k_n^{1-\delta}\big)^2}{\big(s^2+k_n^2\big)^{2+\delta}} ds \\ + \int_1^\infty \frac{s^2\big(1+k_n^{1-\delta}\big)^2}{\big(s^2+k_n^2\big)^2} ds \Bigg)^{1/2}. \end{aligned} \tag{109}$$

where $C := \int d^3\eta' \big|V_{13}(\alpha'\eta')\big|(1+|\eta'|)^{2\delta}$ is finite by (17). The last integral in (109) is clearly uniformly bounded for all n. To see that the first integral in (109) is uniformly bounded we use the following inequality

$$(s^{1-\delta} + k_n^{1-\delta})^2 \leq 2(s^{1-\delta})^2 + 2(k_n^{1-\delta})^2 \leq 4(s^2+k_n^2)^{1-\delta}, \tag{110}$$

where we used $a^\alpha + b^\alpha \leq 2(a+b)^\alpha$ for any $a, b \geq 0$ and $0 \leq \alpha \leq 1$. Hence,

$$\int_0^1 \frac{s^2\big(s^{1-\delta}+k_n^{1-\delta}\big)^2}{\big(s^2+k_n^2\big)^{2+\delta}} ds \leq 4\int_0^1 \frac{s^2 ds}{\big(s^2+k_n^2\big)^{1+2\delta}} \leq 4\int_0^1 \frac{s^2}{s^{2+4\delta}} ds \leq 8. \tag{111}$$

Thus the rhs of (109) goes to zero by Lemma 2. □

Lemma 5. *The following estimates hold*

$$J(\eta', p_\zeta) \leq \frac{c(1+|\eta'|)^\delta}{p_\zeta^2 + k_n^2} \qquad \text{for} \quad |p_\zeta| \geq 1, \tag{112}$$

$$J(\eta', p_\zeta) \leq \frac{c(1+|\eta'|)^\delta}{\big(p_\zeta^2 + k_n^2\big)^{1+\delta/2}} \qquad \text{for} \quad |p_\zeta| \leq 1, \tag{113}$$

where $c > 0$ is a constant.

Proof. Using the trivial inequality $|z+z'|^\delta \leq |z|^\delta + |z'|^\delta$ for any $z, z' \in \mathbb{R}^3$ it is easy to see that

$$\int d^3\zeta \; \Big(1 + \big|m_{y\eta}\eta + m_{y\zeta}\zeta\big|^\delta\Big) e^{-\tilde{b}_2|m_{x\eta}\eta+m_{x\zeta}\zeta|} \leq c'(1+|\eta|)^\delta, \tag{114}$$

where $c' > 0$ is some constant. Using (107) and (114) we obtain

$$J(\eta', p_\zeta) \leq c' \int d^3\eta \, \frac{e^{-\sqrt{p_\zeta^2+k_n^2}|\eta-\eta'|}}{|\eta-\eta'|}(1+|\eta|)^\delta \tag{115}$$

$$\leq c' \int d^3t \, \frac{e^{-\sqrt{p_\zeta^2+k_n^2}|t|}}{|t|}(1+|t+\eta'|)^\delta \leq c' \int d^3t \, \frac{e^{-\sqrt{p_\zeta^2+k_n^2}|t|}}{|t|}\big\{1+|\eta'|^\delta+|t|^\delta\big\}. \tag{116}$$

Now the statement easily follows. □

Physical Remark. In nuclear physics one encounters nuclei [14], which effectively possess the three–particle Borromean structure consisting of two neutrons and a core (Borromean means that the three constituents are pairwise unbound rather like heraldic Borromean rings). The ground states in some of these nuclei are weakly bound and two neutrons form a dilute halo around the core. The calculated correlation plots in [14] reveal the formation of the so–called "dineutron peak" in the ground state probability density, which is well fitted by (4).

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